Quantum Game Theory

Sean Curtis, Christian Thomassy, Cody Lacey

Department of Computer Science

Kennesaw State University
Marietta, Georgia

scurti19@students.kennesaw.edu, cthom417@students.kennesaw.edu, clacey3@students.kennesaw.edu

ABSTRACT

Quantum computing is a computing paradigm that utilizes the properties of quantum mechanics such as superposition, interface and entanglement for data processing and other tasks. Quantum computing can be used to work on the same problems existing supercomputers do but in a much more efficient manner. Classical game theory is a process of modelling that is widely used in AI applications. The extension of this theory to the quantum field is known as quantum game theory. It can be a promising tool for overcoming critical problems in quantum communication and the implementation of quantum artificial intelligence. Quantum game theory allows the player(s) to formulate strategies differing from the conventional way of playing a game. We review the performance of classical and quantum strategies on five classical games by analysing the logic and outcomes of each approach.

Introduction

Game theory is a field of math used to analyse strategies of two or more complying participants. Game theory was limited to only a classical approach, but the development of quantum mechanics and quantum computing have provided an alternative approach. This approach allows players to perform quantum moves producing outcomes that classical game theory would render impossible. This method is known as quantum game theory, which utilizes properties of quantum mechanics such as superposition, interference, and entanglement. The way to implement those properties in quantum computing is by using quantum bits, better known as qubits. They represent information similar to the fundamental bit of a conventional computer. However, single qubits can represent two normal bits of information by simultaneously representing a 0 or 1[9]. Therefore, two qubits can represent four normal bits of information. By following this logic, x number of qubits can be equated to 2^x bits. This is the fundamental advantage quantum computing has over classical computing. The tools we used for our research involved Tensorflow and Google Cirq[6-8].

I. Classical Games

In this section we layout the classical rules behind the five games tested as well as describe the statistical representation of possible outcomes relative to those classical rules.

A. Coin-Flip

The classic coin flip game is a game that can be represented an event producing a binary set of outcomes, heads or tails. Alice and Bob are the players and Bob is trying to guess which side of the coin Alice has facing up. The game is set up where Alice holds a coin with heads facing up. Bob has the first move to decide to flip the coin or keep its state. The second move is for Alice where she can do the same without knowledge of Bob's move. Lastly, Bob gets the final turn with the same possible actions. Bob wins if the coin the heads side facing up. With the classical approach, Alice and Bob each have an equal chance of winning. Alice and Bob have a 50/50 chance of winning.

It can be noted that Alice and Bob are acting on a single bit of data from a computer standpoint. That bit is the only object being influenced in this game.

B. Prisoner's Dilemma

Prisoner's Dilemma is a classic game where Alice and Bob are in prison for a crime and have the choice to confess or stay silent. The two players can decide to confess even though silence provides the optimal outcome. Alice and Bob are isolated and questioned by the police such that neither player knows what choice the other makes. If they both choose to stay silent, they will be given an equally short sentence which is the optimal solution. If Alice stays silent and Bob confesses, or vice versa, the one who confesses will be released and the other will receive the longest sentence of ten years. If Alice and Bob both confess, they will both receive a longer sentence, but not as long as they would if they were both silent[5].

TABLE I
Prisoner's Dilemma

	Bob is Silent	Bob Confesses
Alice is Silent	(2 years, 2 years)	(10 years, 0 years)
Alice Confesses	(0 years,10 years)	(4 years, 4 years)

Fig. 1. Possible outcomes of Alice and Bob's choices

In terms of bits of information in a computer, Alice and Bob each have their own bit to manipulate which allows for more possible outcomes than the coin-flip game.

C. Monty Hall

The Monty Hall game a game where there are three doors and choosing the correct door wins the game. Alice will select the door she thinks hides the prize behind it. Then, the game administrator Bob, will open a door that does not have the prize behind it. Alice now has the option to change doors, she can either choose the unopened door or stick with the door she first selected. Classical game theory proves switching doors gives Alice a 66% chance of winning rather than staying with the initial door, which only gives her a 33% of winning.

TABLE II
Monty Hall

Alice's Choice	Prize Location	Bob's Opened Door(s)	Result if Alice Stays	Result if Alice Switches
1	1	2 or 3	Win	Lose
1	2	3	Lose	Win
1	3	2	Lose	Win
2	2	1 or 3	Win	Lose
2	1	3	Lose	Win
2	3	1	Lose	Win
3	3	1 or 2	Win	Lose
3	1	2	Lose	Win
3	2	1	Lose	Win
		Win %	33%	66%

Fig. 2. Possible outcomes of Monty Hall where Alice tries guessing if the prize is behind door # 1, 2, or 3.

If this game was represented as bits, Alice would be guessing which bit is a 1 while the other two bits are 0's. Bob can only show which bit is a 0. This means Alice will win if she chooses to switch 2/3 of the times.

D. Tick-Tack Toe

Tic Tac Toe is a classic game where two players take turns placing X's and O's on a 3x3 grid. The goal is to get three in a row either vertically, diagonally, or horizontally. The game ends in a draw if no player achieves this goal.

The rules are that two players are competing against each other, and one player is using X's and the other is using O's. The player with X's gets to go first and then the player with O's follows. The question game theory asks is, what are the possible unique outcomes. Unique meaning the boards are not equivalent to other boards even if they are rotated or reflected. There are 138 unique possibilities, and the explanation is as follows. The possible number of board layouts using blanks, X's, and O's is 19,638. This leads to 362,880 possible games (which is equivalent to 9!). The rules of the game greatly shrink the possible boards layouts because X must go first and O follows. Therefore, boards full of X's will never exist. The game ends when three-in-a-row is achieved or there are no more spaces to fill which results in a draw. The outcomes calculated from this are 91 winning outcomes for X, 44 for O, and 3 positions are draws. By excluding solutions that can be replicated via rotation or reflection, there are 138 unique outcomes.

III. Quantum Game Theory

Now that the games have been described in their classical fashion, we will describe how the outcomes differ when using quantum mechanics. When implementing quantum mechanics, bits are now represented as qubits. We will show how the outcomes change with the use of quantum mechanics.

A. Quantum Coin Flip

The coinflip game is now played with the same rules, but Bob can use quantum mechanics and Alice will remain using classical strategies. The game is played with a single qubit; therefore, qubit may be put in a superposition during this game.

The coin is to be considered a single qubit in the quantum realm. Therefore, the coin's sides are labelled as the following: $|0\rangle$ is heads and $|1\rangle$ is tails. The vector representation of each side is

$$\text{Heads}\rightarrow|0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{Tails}\rightarrow|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

The vector representation of the coin is

$$|\text{coin}\rangle = v_0|0\rangle + v_1|1\rangle \rightarrow \begin{bmatrix}v0\\v1\end{bmatrix}$$

Since Alice is holding the coin heads up to start, the initial state of the coin is 0. Bob is now able to make his first move using quantum mechanics. Bob does this by applying the Hadamard operator to the state of the coin[1].

$$\text{Hadamard Gate(H)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

The operation can be written out as

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Bob has completed his turn and next is Alice's turn. She plays a classical move on the coin, which does not change the state of the coin. Lastly, Bob plays the Hadamard operator again resulting in heads as the final state.

$$H(H|0\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\0\end{pmatrix} = \text{Heads}$$

Because Bob has the ability to use quantum mechanics and Alice is limited to classical moves. The outcome of the game always results in Bob winning.

B. Quantum Prisoner's Dilemma

Unlike the coin flip game, Prisoner's Dilemma will be able to use quantum mechanics through a two qubit system. This allows gives us the opportunity to entangle the qubits if necessary. The is set up where both prisoners are given a qubit. With these qubits, they can each apply a single-qubit unitary strategy. Their strategies will be labelled as U1 and U2. The game starts with 2 |0⟩ qubits. There are also 2 classical bits used to store the outcome. The qubits are entangled by an operator prior to a either prisoner making a decision. The operator is labelled J giving the equation J|00⟩. Figure 3 provides the equation for the J operator.

$$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes N} + i\hat{\sigma}_x^{\otimes N})$$

Fig. 3. J-Gate Equation

Let it be noted that the decisions from the prisoners will be represented as U1U2J|00⟩ since their actions are only relevant once the qubits are in an entangled state. Once the prisoners make their individual decision, the qubits are put through a disentanglement gate (conjugate of entanglement gate) J† ending with J†U1U2J|00⟩. Lastly, the qubits must be measured which will cause them to decohere, but it will produce the appropriate outcome. The outcome is a combination of the two classical bits where the payoff can be calculated.

C. Quantum Monty Hall

In this game of Monty Hall, Alice and Bob will be the players and Alice will have quantum moves and Bob will be restricted to only classical. Alice's set of rules will be that she can't open the door that Bob chooses, and she is not allowed to open the door with the prize in it. In this scenario, Bob will act as a normal contestant on the game show and his probabilities of winning remain the same as the classical approach, but there is some differentiation in finding that probability with the quantum approach.

When playing this game using quantum mechanics, we must represent the three doors as a superposition of three equal states. Assume that the doors are labelled 0 through 2. Door 0 would be represented as |100⟩, Door 1 would be represented as |010⟩, and Door 2 would be represented as |001⟩ where 1 is the location of the door containing the prize. We would like the state of the qubits to be in a superposition. Figure 3 represents the Kronecker product of three qubits states [3].

$$|cba\rangle = \begin{bmatrix} c_0b_0a_0 \\ c_0b_0a_1 \\ c_0b_1a_0 \\ c_0b_1a_1 \\ c_1b_0a_0 \\ c_1b_0a_1 \\ c_1b_1a_0 \\ c_1b_1a_1 \end{bmatrix}$$

Fig. 3. Kronecker product of three qubits.

If the states of the qubits are equally superposition then there is a 33% percent chance of the prize being behind each door. A test was done using CNOT gates where a fourth qubit was used to measure the probability of each door having the prize behind it[10]. The outcomes turned out to be the same as if it were done classically. The representation of a three qubit system is what Figure 4 describes. It is labelled as a W gate [4].

$$|W\rangle = 1/\sqrt{3}\ (|001\rangle + |010\rangle + |100\rangle)$$

Fig. 4. Entanglement state of a three qubit system

Now that the game is set up, it is Bob's turn to simply choose a door. Let's say he chose Door 0. Alice will then use quantum mechanics to decide how to proceed. She will use her own qubit to decide which door to open. The two states of her qubit will match the action of opening Door 1 or Door 2. Let's say Door 1 represents the |0⟩ state and Door 2 the |1⟩ state. Assume that the state of the qubit is initialized to the 0 state. If the prize was behind Door 1 then Alice would have to change the state of her qubit to 1 because she would be forced to open Door 2. On the other hand, if the prize was behind Door 2 the state of her qubit would remain unchanged because it would hold the 0 state already. Since Bob chose Door 0, she is no longer limited to one choice. Therefore, she can place her qubit in a superposition where Door 0 is in a 1 state. This equates to four possible scenarios. The probably of those scenarios in terms of switching or staying equate to that of a classical approach.

The game can hold all possibilities at once in this state. However, if the state of the qubit is to be measured then there will be only one door with the prize behind it. Measurements are done partly through the game Paradoxically, we need to measure where the prize is in order to optimize the quantum mechanics for this game.

D. Quantum Tic-Tac-Toe

Now that the basis of classical Tic-Tac-Toe has been determined, the quantum strategy is fairly different. Similar to how the classical version starts, every player will take turns placing into boxes, however, there will be a placement of two particles for each turn a player makes. These particles will be entangled where either one could be transformed into the given player's mark; however, it will not be certain until the quantum particles "collapse". Some of these may also share a box; but, like previously mentioned, it will not be certain until the quantum particles collapse. As the game plays out, some of the particles will form an entanglement loop. In this scenario, the quantum particles "collapse" where the particles are forced outside of its current box if it shares with other particles and force them into empty, adjacent boxes. Once the collapse begins, there are two different ways it can be completed which is determined by the person who did not establish the loop originally, giving one player an advantage over the other. This will continue until one of four outcomes are achieved: One of the two players win, a draw where neither player win, or, unlike classical, both players win.

The question to answer next is, what happens if one player can utilize quantum strategies against another player in a classical game of Tic-Tac-Toe? Let's assume that player O is the initial player and player X is the secondary player. Let's also assume O can use quantum strategies while X cannot. O's starting move will be a classical O in the middle, however, O will also place two quantum bits in box 6 and 8 on this turn. Now, X has to perform their turn. X is now in a position where it has to choose between box 2 or box 4, however because they cannot use quantum strategies, they are left with a clear disadvantage where O can easily win the game. While victory is not certain for O, they are at a much greater advantage than that of X.

IV. Experiment

We created program simulating the use of quantum mechanics and qubits on three of the five classical games.

A. Coin-Flip

The game was tested where Alice and Bob were limited to classical moves, then it was played where Bob could quantize the state of the coin.

We ran a program that simulated the coin-flip game with one player using quantum moves. The program had two test cases that were run 1000 times each. Fig. 5 shows the results if the player using quantum moves decided to flip the coin.

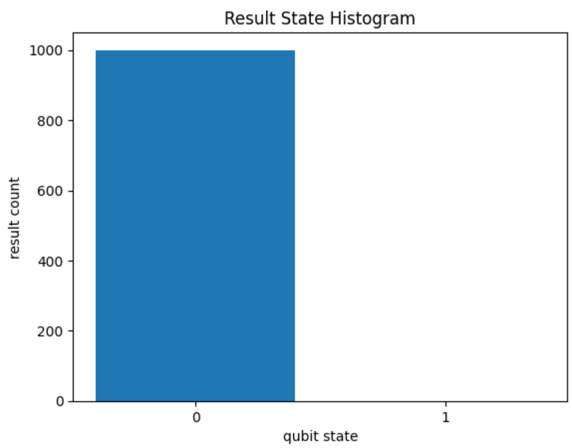

Fig. 5. Results if the quantum player decides to flip the coin

Fig. 6 shows the result if that player decides not to flip the coin.

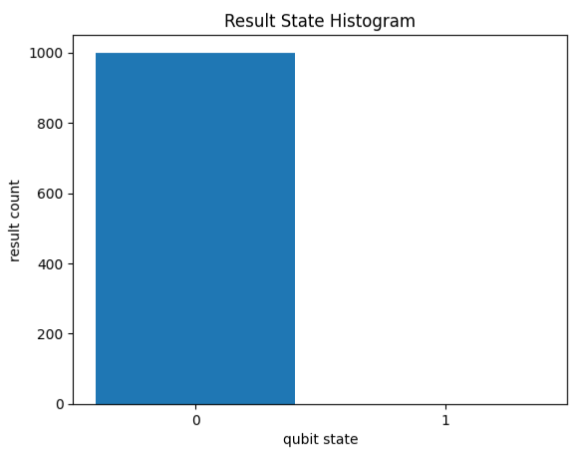

Fig. 6. Results if the quantum player decides not to flip the coin

The tests show the player with quantum moves will always win if they guess heads because the coin will always be in the heads position. The probability that the player with quantum moves will win if they guess heads is 100%.

B. Prisoner's Dilemma

This game was tested where Alice and Bob committed a crime and decide whether to confess or not.

We ran a program 1000 times that simulated the prisoner's dilemma where Alice and Bob's moves are done in an entangled state. Figure 7 shows the outcomes associated with the numbers on the results which are shown by Figure 8.

Prisoner's Dilemma

	Bob is Silent	Bob Confesses
Alice is Silent	(2 years, 2 years) Option 0	(10 years, 0 years) Option 1
Alice Confesses	(0 years,10 years) Option 2	(4 years, 4 years) Option 3

Fig. 7. Decision table of Alice and Bob's choices

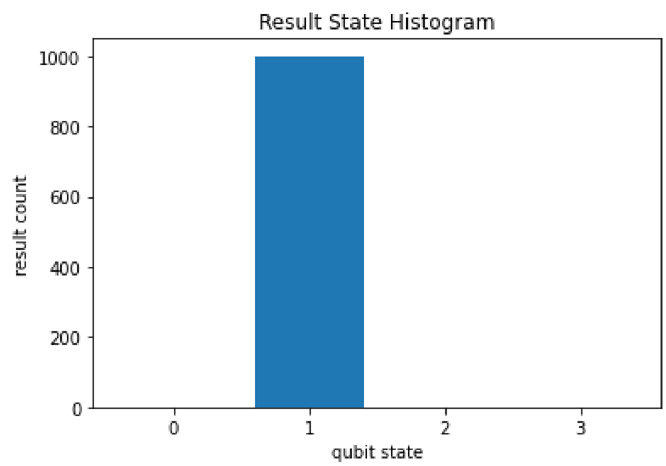

Fig. 8. Results from the prisoner's quantum decisions

Outcome #1 was the option Bob chose to get the optimal outcome because no matter the choice Alice makes, confessing is the optimal outcome.

C. Monty Hall

When running the test 1000 times it shows that if the contestant stays, they will win about 50% of the time and if they change will win about 69% of the time. This is very similar to the classical Monty Hall problem where it is better to change your door than to stay. However, staying in this version of the game is slightly better than the 1/3 chance in the classical version of the game.

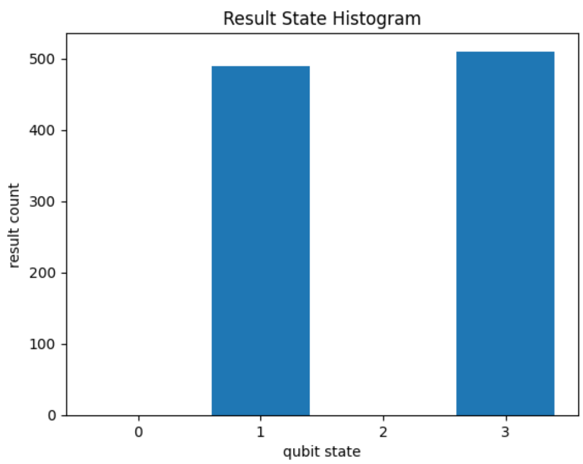

Fig. 4. Results if the quantum player decides not to flip the coin

The outcomes produced by the output state the chance to win is 50% if Bob stays and 69% if he switched.

V. Conclusion

This project investigated the potential of quantum computing in game theory through analyzing the performance of selected games such as coinflip, prisoner's dilemma, survival of the fittest, Monty Hall, and tic tac toe, the project demonstrated the advantages of quantum computing in game theory, particularly in the aspects of entanglement and superposition. The findings of the project showed that quantum strategies can provide players with a significant advantage over classical strategies in certain games. The project also highlighted the limitations of quantum computing in game theory, in that not every game is changed by applying quantum. Overall, this project provides insights into the strengths and limitations of quantum computing in game theory.

VI. Acknowledgements

We would like to thank our project sponsor, Dr. Yong Shi, for guiding us and providing the resources for this team to learn and work. Dr. Shi was instrumental in advising us on which direction to take this research project.

We would also like to thank our project advisor, Sharon Perry, for providing us with the resources and guidance needed. She helped us organize and prepare for each milestone.

References

[1] Adrian P. Flitney and D. Abbott, "An introduction to quantum game theory - researchgate." https://www.researchgate.net/profile/Adrian-Flitney/publication/2188614_An_introduction_to_quantum_game_theory/links/0fcfd50999fefca27e000000/An-introduction-to-quantum-game-theory.pdf?origin=publication_detail.

[2] F. S. Khan, N. Solmeyer, R. Balu, and T. S. Humble, "Quantum Games: A review of the history, current state, and interpretation," *Quantum Information Processing*, 03-Oct-2018. https://www.osti.gov/servlets/purl/1513429.

[3] J. Wright, "Lecture 2: Quantum math basics 1 complex numbers." https://www.cs.cmu.edu/~odonnell/quantum15/lecture02.pdf.

[4] https://iopscience.iop.org/article/10.1088/1742-6596/532/1/012003/pdf

[5] S. E. Landsburg, "Quantum Game theory," *arXiv.org*, 28-Oct-2011. [Online]. Available: https://arxiv.org/abs/1110.6237. [Accessed: 24-Apr-2023].

[6] https://www.tensorflow.org/

[7] https://quantumai.google/cirq

[8] https://medium.com/@patrick.huembeli/hello-world-in-pennylane-and-tensorflow-quantum-compared-8845b9223546#:~:text=Since%20Tensorflow%20Quantum%20%28tfq%29%20does%20all%20the%20quantum,to%20choose%20which%20device%20we%20want%20to%20use

[9] "In-depth guide to Quantum Artificial Intelligence in 2023," *AIMultiple*. https://research.aimultiple.com/quantum-ai/.

[10] M. V. Olivella, "The quantum monty hall problem - diposit.ub.edu." https://diposit.ub.edu/dspace/bitstream/2445/140429/1/VENTURA%20OLIVELLA%20Marc.pdf.